Box 1.1 Unrestricted Growth: Two Different Approaches When the Rate of Growth Is Constant

Discrete Time Using a Difference Equation (Geometric Growth)

The starting expression is

$$N_{t+1} = RN_t + N_t,$$

where R is the net discrete (or geometric) per capita rate of growth. Collecting terms gives

$$N_{t+1} = (R + 1)\,N_t, \qquad (1.1a)$$

which we may also write as

$$N_{t+1} = \lambda N_t,$$

where $\lambda = (R + 1)$ is the discrete (or geometric) per capita rate of growth; its units are per time period.

Similarly, for two time steps,

$$N_{t+2} = \lambda N_{t+1}$$
$$= \lambda\lambda N_t$$
$$= \lambda^2 N_t$$

and, for any arbitrary number of time steps into the future (say, T time steps),

$$N_{t+T} = \lambda^T N_t. \qquad (1.2a)$$

If we start with N_0 individuals at time $t = 0$, then at time T the number of individuals is

$$\mathbf{N_T = N_0 \lambda^T}. \qquad \mathbf{(1.3a)}$$

Equation (1.3a) is the "solution" for discrete time because it is a formula giving N for any arbitrary time period into the future.

Continuous Time Using a Differential Equation (Exponential Growth)

The starting expression is

$$\frac{dN}{dt} = rN(t). \qquad (1.1b)$$

Initial conditions specify the beginning time (t = 0) and initial population size

$$N\ (\text{at } t = 0) = N(0).$$

In Eq. (1.1b), r is the intrinsic (or exponential) per capita rate of growth; its units are per time period.

To solve Eq. (1.1b) with its initial conditions, we separate the differentials and integrate both sides. Then we evaluate the integral from $t = 0$ to $t = T$:

$$\int_{N(0)}^{N(T)} \frac{dN(t)}{N} = r\int_0^T dt. \qquad (1.2b)$$

From the integral formulas of calculus, the left-hand side of Eq. (1.2b) becomes

$$\ln N(T) - \ln N(0)$$

and the right-hand side of Eq. (1.2b) becomes

$$rT - r0 = rT.$$

After exponentiation of both sides,

$$\frac{N(T)}{N(0)} = e^{rT}.$$

Finally, rearranging yields

$$\mathbf{N(T) = N(0)\, e^{rT}}. \qquad \mathbf{(1.3b)}$$

Equation (1.3b) is the "solution" for continuous time because it is a formula giving $N(t)$ for any arbitrary time T into the future.

Comparing Eqs. (1.3a) and (1.3b), we see that $e^r = \lambda$.